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AVERAGE MARKS SCALING

§1. Introduction

In Western Australia a wide range of courses with an external examination taken at the end of Year 12 are available for senior secondary students. The examination results are then combined with school assessments. This process involves *moderation* of school assessments using the raw examination marks. Within each course, the combined marks are standardised[1].

Thus, for each student a composite unscaled mark[2] is generated for each course taken. However, the cohort of students taking a particular course P may be academically more able than the cohort of students taking course Q. This means that a composite unscaled mark of, say, 60 in course P represents a higher level of attainment than the same mark in course Q. Since marks in different courses are added to form a Tertiary Entrance Aggregate (TEA), equity considerations require that the marks in courses P and Q should be *scaled*. This means that the marks in course P should be scaled up relative to the marks in course Q (or the marks in course Q scaled down.) Of course, the situation is more complex than just establishing the appropriate relativity of the marks in courses P and Q. In fact there are many courses that students can take and so it is the relativities of all courses which scaling needs to address.

The key concept is to generate a measure of academic ability, based on students' achievements, for the cohort of students taking a particular course. The scaling method used in Western Australia[3] is based on the premise that the best measure of an individual student's academic performance is that student's average scaled score across all courses taken. Suppose that average is denoted by t_i. That is to say, for a particular student i define t_i to be the *average scaled score* obtained by student i across all courses taken by student i.

[1] Prior to 2016, the process involved moderating the school assessments against standardised examination marks.
[2] This is often referred to as the final combined mark.
[3] Average Marks Scaling has been used in New South Wales for many years. The main architect of that system was Professor E Senata, Department of Mathematical Statistics, University of Sydney. The New South Wales system was adapted for use in Western Australia by Dr M.T. Partis, Director, Secondary Education Authority, in 1997 and introduced in 1998. Prior to that the Australian Scaling Test was used as the anchor variable in the scaling process.

Now consider the cohort of students taking a particular course j. For *each* of these students there will be an average scaled score t_i. The average of the t_i across *all* the students taking course j will be denoted by T_j.

That is to say, for a particular course j define T_j to be the *average* of the *average scaled scores* t_i obtained by all students taking course j. The Average Marks Scaling (AMS) process uses T_j as a proxy measure[4] of the academic ability of the cohort of students taking course j.

At the end of the process the value of T_P for course P and the value of T_Q for course Q can be calculated. The mean[5] mark for the cohort of students taking course P is then scaled to T_P, whilst the mean mark for the cohort of students taking course Q is scaled to T_Q. This gives the appropriate relativity between the two courses. Adjustments are also made to the standard deviations of the course distributions, but these turn out to be minor compared to the adjustments to the means.

It is worth stressing the main feature of Average Marks Scaling. The description above seems to suggest that the scaled scores need to be known in order for scaling to be carried out. However, at the heart of the process is the equating of the average scaled score in course j and the anchor variable T_j. The mathematics involved is set out below, but the key equation

Average scaled score in course j of the cohort of students taking course j = The anchor variable T_j for the course j …… (1)

provides the central focus of the analysis.

To clarify the concepts involved it is worth considering a numerical example. Suppose that student i has a scaled score of 63 in course j. In what follows this scaled score will be denoted by y_{ij}.

That is to say, $$y_{ij} = 63$$

Suppose that student i takes four other courses obtaining scaled scores in those of 47, 71, 58 and 66. Then the average of this student's five scaled scores will be 61.

That is to say, $$t_i = 61$$

It is important to note that, in this instance, the values of y_{ij} and t_i are not the same. This is because they are measuring different things. y_{ij} is measuring the student's performance in course j, whilst t_i is measuring the student's performance across five courses.

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4 T_j is sometimes described as the *anchor variable* for the scaling process.

5 Throughout this document the terms 'mean' and 'average" will be regarded as synonymous.

Now consider the cohort of all students taking course j. For each of these students there will be corresponding values of y_{ij} and t_i. Hence, the average values of y_{ij} and t_i for the whole cohort can be determined. The scaling process is designed to make these averages identical. In simple terms the unscaled marks in course j are moved up or down to achieve this outcome.

The adjustment of the composite unscaled marks to scaled scores uses a linear conversion, the details of which are explained below. It is important to note the following points:

- the AMS scaling process preserves the ranking of students and the shape of the distribution in each course;
- the mean scaled score across all courses and all students (the *global mean*) is predetermined: set at 60.
- the standard deviation parameter for the AMS process is predetermined: set at 14.

§2. Re-standardisation using z-scores

Although examination marks and school assessments are standardised at an earlier stage of the process, it is mathematically convenient to standardise again in such a way that the composite unscaled mark distribution for each course has a mean of 0 and a standard deviation of 1.

Let w_{ij} be the composite unscaled mark for student i in course j. For each course j let μ_j and σ_j be the mean and standard deviation of the w_{ij}.

Now put

$$z_{ij} = \frac{w_{ij} - \mu_{ij}}{\sigma_j}.$$

The z_{ij} values generated in this way are often referred to as z-scores. By definition it follows that the z-scores for each course will have a mean of 0 and a standard deviation of 1.

The AMS process uses the z_{ij} values defined above to generate scaled scores y_{ij}. The conversion to scaled scores can be regarded as a three-part process.

First, add 60 to restore the overall mark distribution to the predetermined global mean.

Second, add a term d_j for each course j which determines whether the marks in that particular course are scaled up or down relative to the global mean. This means that some d_j will be positive and some negative.

Third, add a term $14c_j z_{ij}$, which alters the standard deviation for each course from 1 to $14c_j$. The value of 14 is predetermined to produce an appropriate global standard deviation. The c_j will be calculated for each course j, but in practice the values turn out to be always close to 1.

The combination of the three steps outlined above gives rise to the following equation:

The scaled score y_{ij} for student i in course j is given by

$$y_{ij} = 60 + d_j + 14c_j z_{ij}. \qquad \text{...... (2)}$$

The parameters d_j and c_j need to be evaluated for each course j. It should be emphasised that whilst equation (2) gives an *algebraic* definition of the scaled scores y_{ij}, the *arithmetical* values of the y_{ij} can only be calculated after the parameters d_j and c_j have been determined. The way in which the parameters d_j and c_j are evaluated is set out below.

§3. Calculating averages

In this section the technical details of working out the average of the average scaled scores, that is to say T_j for each course j, are developed. For this purpose it is useful to introduce a function α_{ij} which depends on whether a particular student is taking a course or not. Define

$$\alpha_{ij} = \begin{cases} 1 & \text{if student } i \text{ takes course } j; \\ 0 & \text{if student } i \text{ does not take course } j. \end{cases}$$

Let n be the total number of students taking the examinations[6].
Let m be the total number of courses available in the examinations.

The number of students n_j taking course j is then given by

$$n_j = \sum_{i=1}^{n} \alpha_{ij}$$

The number of courses m_i taken by student i is given by

$$m_i = \sum_{j=1}^{m} \alpha_{ij}$$

For a particular student i the *average scaled score*, denoted by t_i, over all courses taken by that student is given by

$$t_i = \frac{1}{m_i} \sum_{k=1}^{m} \alpha_{ik} y_{ik}$$

[6] In practice a subset of the total number of students, known as the *scaling population*, is used. The intention is to exclude, for example, students taking only one course.

For all the students taking a particular course j the *average* of the *average scaled scores*, denoted by

$$\begin{aligned} T_j &= \frac{1}{n_j} \sum_{i=1}^{n} \alpha_{ij} t_i \\ &= \frac{1}{n_j} \sum_{i=1}^{n} \alpha_{ij} \left\{ \frac{1}{m_i} \sum_{k=1}^{m} \alpha_{ik} y_{ik} \right\} \end{aligned}$$

$$= \frac{1}{n_j} \sum_{i=1}^{n} \alpha_{ij} \left\{ \frac{1}{m_i} \sum_{k=1}^{m} \alpha_{ik} \left(60 + d_k + 14 c_k z_{ik} \right) \right\}$$

The significance of t_i and T_j was explained in §1. T_j is the anchor variable for the scaling process.

§4. Matrix representation

In §3 the formula for T_j was derived. This generates m equations, corresponding to the values of j running from 1 through to m. The next stage in the process is to recast these m equations in a matrix format.

From the previous section it follows that

$$T_j = \sum_{k=1}^{m} \left(60 b_{jk} + d_k b_{jk} + 14 c_k a_{jk} \right)$$

where

$$b_{jk} = \frac{1}{n_j} \sum_{i=1}^{n} \frac{\alpha_{ij} \alpha_{ik}}{m_i} \quad \text{and} \quad a_{jk} = \frac{1}{n_j} \sum_{i=1}^{n} \frac{\alpha_{ij} \alpha_{ik} z_{ik}}{m_i}$$

This leads to the matrix equation

$$\underset{\sim}{T} = 60 B \underset{\sim}{1} + B \underset{\sim}{d} + 14 A \underset{\sim}{c} \qquad \text{..... (3)}$$

where

$\underset{\sim}{T}$ is the $m \times 1$ column vector $[T_j]$;

A and B are the $m \times m$ matrices $[a_{jk}]$ and $[b_{jk}]$, respectively;

$\underset{\sim}{1}$ is the $m \times 1$ column vector with each entry equal to 1;

and $\underset{\sim}{c}$ and $\underset{\sim}{d}$ are the $m \times 1$ column vectors $[c_j]$ and $[d_j]$, respectively.

Analysis of the b_{jk} which are the elements of matrix B then gives

$$B \underset{\sim}{1} = \underset{\sim}{1}$$

Hence, equation (3) simplifies to

$$\underset{\sim}{T} = 60 \underset{\sim}{1} + B \underset{\sim}{d} + 14 A \underset{\sim}{c} \qquad \text{..... (4)}$$

§5. Average scaled score in course *j*

From equation (2) the scaled score y_{ij} for student i in course j is given by

$$y_{ij} = 60 + d_j + 14c_j z_{ij}.$$

Hence, for a particular course j, the average of the y_{ij} is given by

$$\begin{aligned}\frac{1}{n_j}\sum_{i=1}^{n}\alpha_{ij}y_{ij} &= \frac{1}{n_j}\sum_{i=1}^{n}\alpha_{ij}(60 + d_j + 14c_j z_{ij}) \\ &= 60 + d_j + 14c_j\left(\frac{1}{n_j}\sum_{i=1}^{n}\alpha_{ij}z_{ij}\right) \\ &= 60 + d_j\end{aligned}$$

since the z-scores for course j have a mean of 0.

From equation (1) this gives

$$T_j = 60 + d_j$$

Hence,

$$\underset{\sim}{T} = 60\,\underset{\sim}{1} + \underset{\sim}{d} \qquad \text{..... (5)}$$

From equations (4) and (5) it follows that

$$60\,\underset{\sim}{1} + \underset{\sim}{d} = 60\,\underset{\sim}{1} + B\,\underset{\sim}{d} + 14A\,\underset{\sim}{c}$$

Simplifying this matrix equation gives

$$(I - B)\,\underset{\sim}{d} = 14A\,\underset{\sim}{c} \qquad \text{..... (6)}$$

where I is the $m \times m$ identity matrix.

§6. Calculation of the key parameters

In matrix equation (6) the only unknowns are the column vectors $\underset{\sim}{c}$ and $\underset{\sim}{d}$. The c_j entries which make up the column vector $\underset{\sim}{c}$ can be thought of as the standard deviations for each course j. This can be evaluated by considering the z-scores for all students taking course j.

The standard deviation c_j for the z-*scores* in course j is given by a complicated (but standard) formula, namely

$$c_j^2 = \frac{1}{n_j}\left\{\sum_{i=1}^{n}\alpha_{ij}\left(\frac{\sum_{k=1}^{m}\alpha_{ik}z_{ik}^2}{\sum_{k=1}^{m}\alpha_{ik}}\right)\right\} - \frac{1}{n_j^2}\left\{\sum_{i=1}^{n}\alpha_{ij}\left(\frac{\sum_{k=1}^{m}\alpha_{ik}z_{ik}}{\sum_{k=1}^{m}\alpha_{ik}}\right)\right\}^2$$

The arithmetical values for the c_j can now be substituted into equation (6). Solving equation (6) gives the values of d_j.

Equation (2) now allows the scaled scores y_{ij} to be calculated for all students in all courses.

§6. Example showing how a student's scaled score is calculated

Consider a student whose unscaled combined mark in course j is 65.73. For the cohort of students taking course j suppose that the mean and standard deviation of the unscaled combined marks are 59.51 and 12.20, respectively.

Now derive the z-score corresponding to the student's unscaled mark of 65.73. This is

$$z_{ij} = \frac{w_{ij} - \mu_j}{\sigma_j} = \frac{65.73 - 59.51}{12.20} = 0.51$$

For course j suppose that the scaling parameters turn out to be c_j = 1.02 and d_j = 5.22.
Then equation (2) gives the scaled score y_{ij} as

$$y_{ij} = 60 + d_j + 14c_j z_{ij} = 60 + 5.22 + 14(1.02)(0.51) = 72.50$$